STUDENTS’ STRUCTURING OF RECTANGULAR ARRAYS

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This paper presents the results of a study of the structural development of young students’ drawings of arrays, and in particular, the significance of using lines instead of drawing individual squares. Students’ array drawings were classified on basis of numerical properties, and perceived structural similarities that reflected the spatial properties of arrays. The relationship between these two aspects was investigated and a sequence for the development of array structure is postulated.

INTRODUCTION

The rectangular array model is important for mathematics learning because of its use to model multiplication, to represent fractions and as the basis for the area formula. Although array models are used to show multiplicative relationships, students may not see structural similarities of discrete arrays and arrays as a grid of contiguous squares, thus they may not connect an array of squares with multiplication.

Fundamental understandings of rectangular array structure would appear to be that the region must be covered by a number of congruent units without overlap or leaving gaps, and that a covering of units can be represented by an array in which rows (and columns) are aligned parallel to the sides of the rectangle, with equal numbers of units in each. The most efficient way of drawing an array is to draw equally-spaced lines parallel to the sides of the rectangle, constructing equal rows and columns. However, many young students cannot do this (Outhred & Mitchelmore, 1992). In this paper we make inferences as to how students’ understandings of array structure progress from a collection of individual units to (perpendicular) intersecting sets of parallel lines.

The action of physically covering a rectangle with unit squares suggests a counting process whereas the array model is used to exemplify multiplication. To link the array model to multiplication, students need to perceive first that the rows are equal and correspond to equivalent groups. In theory, such a perception equates to a repeated addition model. The second perception, that the array is a composite of composites, equates to a multiplicative model. Steffe (1992) believes that students’ recognition and production of composite units are key understandings in learning about multiplication. However, students may not fully understand the relationship between multiplication and addition (Mulligan & Mitchelmore, 1997) and may persist in counting.

Only gradually do students learn that the number of units in a rectangular array can be calculated from the number of units in each row and column (Battista, Clements, Arnoff, Battista, & Borrow, 1998). These authors classified Grade 2 students’ counting methods into levels of increasing sophistication. At the lowest level, students counted in a disorganised manner. Then there was what Battista et al. call a paradigm shift to treating the array in terms of rows. Some students were unsure of how to find the number of rows, while others were able to find the number of rows when the number of squares in the orthogonal direction was given but estimated this number otherwise. By contrast, at the highest strategy level students immediately used the numbers of units in each row and column to find the total by multiplication or repeated addition.

In area measurement, emphasis on area as covering encourages counting (Hirstein, Lamb & Osborne, 1978; Outhred & Mitchelmore, 2000). Students may not see congruence as crucial to measurement; they may perceive individual pieces resulting from partitioning regions as counting units rather than fractional portions of a referent whole (Hirstein et al., 1978; Mack, 2001). Students who count units are also unlikely to link area measurement to multiplication, which is fundamental to understanding the area formula. Use of concrete materials also encourages counting and does nothing to promote multiplicative structure. The materials themselves can obscure structural features of unit coverings (Doig & Cheeseman, 1995; Dickson, 1989) and obviate the need for students to structure arrays.

Student's drawings of arrays

How do students develop mental representations of array structure by abstraction from physical or pictorial models? There is evidence that students' drawings of array models can reveal their mental representations (Besur & Eliot; 1993). We shall assume that students' difficulties in representing arrays are a consequence of limited conceptions of array structure, rather than of inadequate drawing skills.

Several researchers (Battista et al., 1998; Outhred & Mitchelmore, 2000) have emphasized the relationship between array structure and either counting or measurement. Neither study focused on development of array representation, nor interpretation of students' drawn constructions in terms of their understanding. No systematic study detailing the structural development of students' array drawings, and in particular, the significance of using lines instead of individual squares, appears to have been reported in the literature.

METHODOLOGY

A large sample of 115 students, with approximately equal numbers of boys and girls from a range of cultural groups, was randomly selected from forty grade 1 to 4 classes (aged 6 to 9 years) in four schools in a medium-socioeconomic area of a large city. Individual interviews with the students were conducted early in the school year.

The sequence of drawing, counting, and measuring tasks involved representing arrays of units given different perceptual cues, calculating the numbers of elements in arrays, and constructing arrays of the correct dimensions when no perceptual cues were given. These particular skills focus on linking the unit (in this case, a square), iteration of this unit to cover a rectangular figure, and the lengths of the sides of the figure. Information concerning the strategies that students used to solve array-based tasks was inferred from students' strategies as they drew. In this paper only a subset of the tasks will be included. These tasks are summarised in Figure 1.

The drawing items (D1, D2, and D3) all required the students to draw arrays of units, but did not require measurement skills. Responses to Task D1 should indicate students' perceptions of the essential features of an array because no drawing cues were given and to copy a figure students require some knowledge of its properties. Tasks D2 and D3 were presented to elucidate students' abilities to construct arrays given different cues, which required that students imagine increasingly more of each array in order to draw it. The responses to the tasks provide information about the skills involved in representing arrays and the order in which these skills are learnt.

Task	Unit	Requirements
D1	Cardboard tile 4cm square	Cover a 12cm x 16cm rectangle (enclosed by a raised border) with 4 cm cardboard unit squares, work out how many units, and draw the squares.
D2	Drawing of a 1cm square	Draw array given units along two adjacent sides of a 4 cm x 6 cm rectangle.
D3	Drawing of a 1cm square	Draw array given marks to indicate the units on each side of a 5 cm x 8 cm rectangle.

Figure 1 **The array drawing tasks (D1, D2, D3)**

RESULTS AND DISCUSSION

The students' drawings were sorted in two ways, based on analysis of the drawings, supplemented by the interview notes. First, the drawings were classified on the numerical properties of arrays, and second, on the basis of perceived structural similarities that reflected the spatial properties of arrays. The numerical classification was based whether students drew equal rows (columns) and whether the dimensions corresponded to the array that had been indicated. The spatial classification was based on covering the region without leaving gaps and the degree of abstraction shown in the drawings, that is whether students drew individual squares or lines. All three tasks showed the same three levels for numerical properties and five levels for spatial properties. However, students often produced drawings at different levels for different tasks, so the levels are not a classification of students. The three final numerical levels are shown in Figure 2.

Level 1. Unequal rows (columns): There may be an incorrect number of columns with an unequal number of units in each (1a) or a correct number of rows with an unequal number of units in each row (1b).

Level 2. Equal rows (columns)—incorrect dimensions: Rows and/or columns have an equal, but incorrect, number of units. There may be an incorrect number of rows with an equal, but incorrect, number of units in each (2a) or a correct number of rows with an equal, but incorrect, number of units in each (2b).

Level 3. Numerically correct array: Rows and columns have an equal and correct numbers of units. However, the array is not always spatially sophisticated.

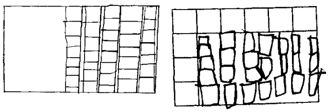
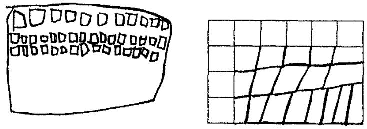
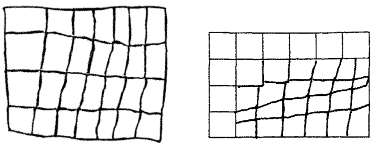
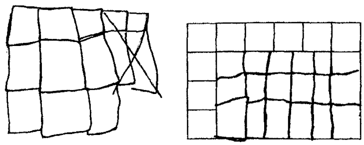
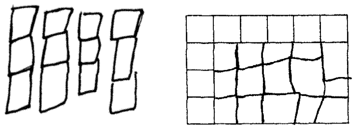
1(a) incorrect number of columns, each with unequal numbers of units 	**1(b)** correct number of rows with an unequal number of units in each. 
2(a) incorrect number of rows with an equal, but incorrect, number of units in each row 	**2(b)** correct number of rows with an equal, but incorrect, number of units in each row 
3 correct number of columns, equal and correct number of units in each 	

Figure 2 **Examples of each numerical level for Tasks D1 and D2**

Spatial structuring levels

The most important skill in representing an array, partitioning into rows and columns, seems to be based on an understanding of a fundamental property of rectangular arrays: the elements of an array are collinear in two directions. The five level classification of spatial structure (see Figure 3 for examples from Task D3) describes students' increasing level of knowledge of array structure from Level 1 to Level 5.

Level 1 Incomplete covering: The units do not cover the whole rectangle. They are drawn individually and may be: (a) unorganised elements; or (b) arranged in one dimension but not connected.

Level 2 Primitive covering: An attempt is made to align units (drawn individually) in two dimensions. Units cover the rectangle without overlap but their organisation is unsystematic.

Level 3 Array covering—Individual units: Units are drawn individually, are approximately equal in size, and are aligned both vertically and horizontally. Drawings show correct structure—equal numbers of approximately rectangular units in each row and column. The array is not constructed by iterating rows.

Level 4 Array covering—Some lines: Students realise that units in rows (or columns) can be connected and use some lines to draw the array.

Level 5 Array covering—All lines: The array is drawn as two (perpendicular) sets of parallel lines. Row iteration is therefore fully exploited.

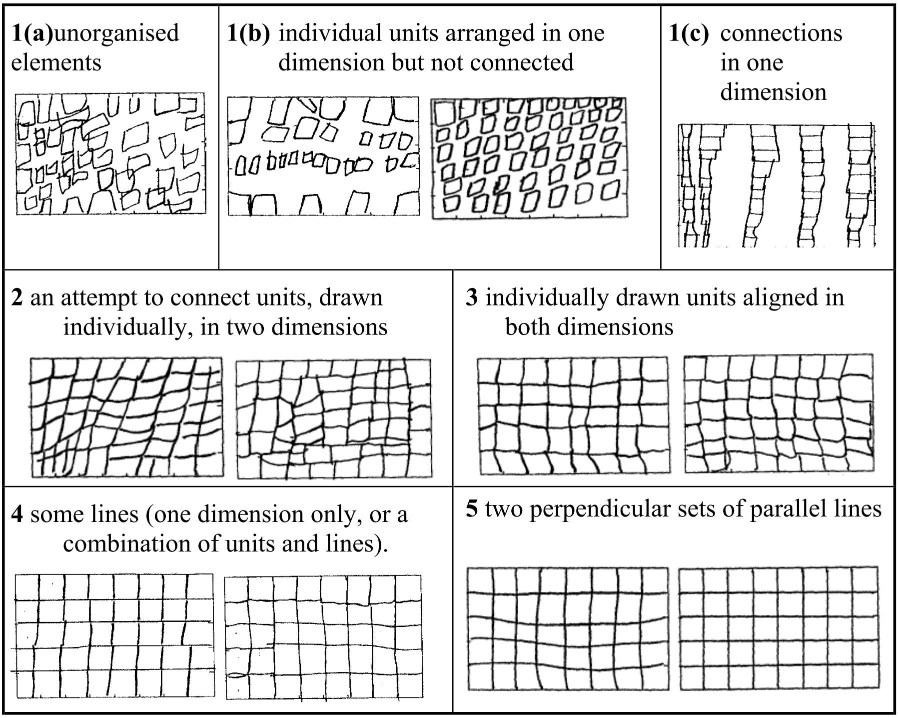


Figure 3 **Examples of each spatial level for Task D3**

The above sequence is developmental in the sense that each level is more sophisticated than the previous ones and the levels show a clear grade progression. This is not to say that students necessarily progress through each level in turn. At Level 1, no discernible strategy is used to cover the rectangle. Young students frequently draw individual units with large gaps between them but as they realise the importance of alignment, their drawings increase in regularity and the row/column structure becomes correspondingly apparent. As student knowledge increases the units become connected first in one, then in two dimensions (that is, students gradually seem to understand the importance of covering the region). Until students attempt to join the units in two dimensions, the rows and columns are not usually aligned. The strategies used to construct coverings at Levels 1, 2 and 3 might be termed local rather than global (Battista & Clements, 1996). The students focus on parts of the structure—for example, iterating rows or joining adjacent squares—but they have no global scheme for coordinating an array.

Level 4 indicates the emergence of a coordinated scheme for showing units as composites in one or two dimensions. There are various transition stages between drawing individual units and an array. For instance, lines may be drawn across the width of the rectangle to indicate rows with units in each row marked off individually, or some individual units (usually the top row and the left column) is drawn as a guide to drawing the array (see Figure 3, Level 4). The most abstract method of drawing an array is as two (perpendicular) sets of parallel lines (Level 5), because this method is furthest removed from the physical action of covering a rectangle with individual units. By Level 5 students appear to have internalized the row and column structure.

The relationship between numerical and spatial levels

The relationship between numerical and spatial levels for Task D3 showed that few students (7%) drew a numerically correct arrangement without using some lines (Levels 4 or 5). The converse was also true, all students' drawings classified as spatial Levels 1 and 2 were numerically incorrect. The distribution of numerical and spatial levels with grade for Task D3 is quite different from Task D1 where, in most of the drawings that did not show a systematic array covering (Levels 1 and 2), rows and columns usually contained unequal numbers of units. Nevertheless, quite a large proportion (37%) of these drawings were numerically correct. Once students began to use lines to draw the array (Levels 4 and 5), they always drew equal numbers of units in each row but,21% of students did not show the correct number of units in each row. However, Task D1, in which numerical structure had to be deduced, would have been far more difficult if the model had had equivalent dimensions to Task D3 (5x8).

CONCLUSION

In summary, the results of this study show students' drawings of rectangular arrays develop between Grades 1 and 4 from single squares to an accurate array with a concomitant understanding of alignment and composite units. Analysis of students' drawings indicated that representing an array of units using two perpendicular sets of parallel lines is more difficult than might be expected, indicating that the structure of a square tessellation is not obvious to students but must be learned.

In initial representations of arrays, many Grade 1 students did not see the importance of joining the units so that there were no gaps, and drew units individually. As they attempted to align squares, their drawings became increasingly regular and the structure became correspondingly apparent. Until students began to join the units in two dimensions, they did not usually align rows and columns. Before drawing arrays using only lines, some students drew lines across the width of the rectangle to indicate rows and marked off the units in each row individually while others drew some individual units (usually the top row and the left column) as a guide to drawing the array. By Grade 4 most students had learnt that the physical action of covering a rectangular area with units was equivalent to an abstract representation using lines.

For some students the lines shown in an array may be only a visual feature unrelated to numerical structure. However, drawing lines in one dimension appeared to be a precursor to recognising rows as composite units. Such recognition helped students to perceive that squares could be constructed by joining lines in the other direction, and hence realise the two-dimensional structure of an array. The comparison between numerical and spatial structure across the three tasks shows that drawing correctly aligned units is necessary, but not sufficient for correct numerical structure when drawing arrays of large dimensions (as in Task D3). Although it might be argued that the relationship is a consequence of the strong correlation between knowledge of array structure and age, students in Grades 2, 3, and 4 solved measurement tasks when the units were not indicated (Outhred & Mitchelmore, 2000), so the more salient discriminator would appear to be array structure. Moreover, for the indicated grid task, numerical level was a stronger predictor of spatial level than grade. The results of this study, combined with a teaching experiment (see Outhred, 1993) suggest the following sequence (see Figure 4) for the development of array structure.

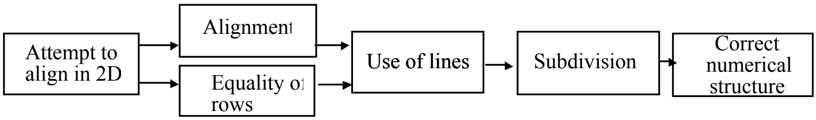


Figure 4 **The hypothesised development of array structure**

Understanding of array structure (as demonstrated by ability to complete an indicated grid) has been shown to be a prerequisite for students to progress from array-based activities with concrete or pictorial support to more abstract tasks, involving multiplication and measurement. Only students who drew an array using at least some lines successfully solved a measurement task in which students had to construct an array of the correct dimensions (5x6) by accurate estimation or by measuring the side lengths of the rectangle with a ruler (Outhred & Mitchelmore, 2000). The results of the measurement tasks reported in the above study reinforced the significance of the formation of an iterable row as the foundation of an understanding of array structure. In addition, an understanding of subdivision was found to be crucial when cues to the array structure were not given. Students have to clearly identify the significance of the relationship between the size of the unit and the dimensions of the rectangle. Although it may seem self-evident to adults that the number of units in the array must depend on the measurements of the sides, it was clearly not obvious to students. Thus, teaching about array structure must include activities that provide students with experience of partitioning a length into equal parts. Subdividing a rectangular region into equal parts depends on students being able to partition a length into a required number of parts, as well as knowing that an array can be represented using lines.

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